Groups in Cryptography

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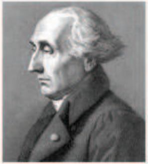
Joseph Louis Lagrange



Joseph-Louis Lagrange, born Giuseppe Luigi Lagrancia was an Italian-born French mathematician and astronomer born in Turin, Piedmont, who lived part of his life in Prussia and part in France. Wikipedia

Born: January 25, 1736, Turin

Died: April 10, 1813, Paris

Education: École Polytechnique

Parents: Maria Theresa Gros, Giuseppe Francesco Lodovico Lagrange

Groups in Cryptography

- A set S and a binary operation $\oplus$ together is called a group $G = (S, \oplus)$ if the operation and the set satisfy the following rules
  - Closure: If $a, b \in S$ then $a \oplus b \in S$
  - Associativity: For $a, b, c \in S$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
  - There exists a neutral element: $e \in S$ such that $a \oplus e = e \oplus a = a$
  - Every element $a \in S$ has an inverse $\text{inv}(a) \in S$:

$$a \oplus \text{inv}(a) = \text{inv}(a) \oplus a = e$$

  - Commutativity: If $a \oplus b = b \oplus a$, then the group G is called an a commutative group or an Abelian group
  - In cryptography we deal with Abelian groups

Multiplicative Groups

- The operation $\oplus$ is a multiplication "$\cdot$"
- The neutral element is generally called the unit element $e = 1$
- Multiplication of an element k times by itself is denoted as

$$a^k = \overbrace{a \cdot a \cdots a}^{k \text{ times}}$$

- The inverse of an element a is denoted as a^{-1}
- Example: $(\mathcal{Z}_n^*, * \bmod n)$; note that $\mathcal{Z}_n^*$ is the set $\{1, 2, \ldots, n-1\}$ when n is prime, and the operation is multiplication mod n
- When n is not a prime, $\mathcal{Z}_n^*$ is the set of invertible elements modulo n, since $a \in \mathcal{Z}_n^*$ implies $\gcd(a, n) = 1$, and thus a is invertible mod n

Multiplicative Group Examples

- Consider the multiplication tables mod 5 and 6, respectively, below

* mod 5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

* mod 6	**1**	2	3	4	**5**
1	**1**	2	3	4	**5**
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	**5**	4	3	2	**1**

- Mod 5 multiplication operation on the set $\mathcal{Z}_5 = \{1, 2, 3, 4\}$ forms the group $\mathcal{Z}_5^*$
- Mod 6 multiplication operation on the set $\mathcal{Z}_6 = \{1, 2, 3, 4, 5\}$ does not form a group since 2, 3 and 4 are not invertible
- Mod 6 multiplication operation on the set of invertible elements forms a group: $(\mathcal{Z}_6^*, * \text{ mod } 6) = (\{1, 5\}, * \text{ mod } 6)$

Additive Groups

- The operation $\oplus$ is an addition "$+$"
- The neutral element is generally called the zero element $e = 0$
- Addition of an element a k times by itself, denoted as

$$[k]\,a = \overbrace{a + \cdots + a}^{k \text{ times}}$$

- The inverse of an element a is denoted as $-a$
- Example: $(\mathcal{Z}_n, + \bmod n)$ is a group; the set is $\mathcal{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and the operation is addition mod n

Additive Group Examples

- Consider the addition tables mod 4 and 5, respectively, below

+ mod 4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

+ mod 5	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

- Mod 4 addition operation on set $\mathcal{Z}_4 = \{0, 1, 2, 3\}$ forms the group $(\mathcal{Z}_4, + \text{ mod } 4)$
- Mod 5 addition operation on set $\mathcal{Z}_5 = \{0, 1, 2, 3, 4\}$ forms the group $(\mathcal{Z}_5, + \text{ mod } 5)$

Order of a Group

- **The order of a group** is the number of elements in the set
- The order of $(\mathcal{Z}_{11}^*, * \bmod 11)$ is 10, since the set $\mathcal{Z}_{11}^*$ has 10 elements: $\{1, 2, \ldots, 10\}$
- The order of group $(\mathcal{Z}_p^*, * \bmod p)$ is equal to $p - 1$; since p is prime, the group order $p - 1$ is not prime
- The order of $(\mathcal{Z}_{11}, + \bmod 11)$ is 11, since the set $\mathcal{Z}_{11}$ has 11 elements: $\{0, 1, 2, \ldots, 10\}$
- The order of $(\mathcal{Z}_n, + \bmod n)$ is n, since the set $\mathcal{Z}_n$ has n elements: $\{0, 1, 2, \ldots, n - 1\}$; here n could be prime or composite

Order of an Element

- **The order of an element** a in a multiplicative group is the smallest integer k such that $a^k = 1$ (where 1 is the unit element of the group)
- order(3) $= 5$ in $(\mathcal{Z}_{11}^*, * \text{ mod } 11)$ since

$$\{\ 3^i \text{ mod } 11 \mid 1 \leq i \leq 10\} = \{3, 9, 5, 4, 1\}$$

- order(2) $= 10$ in $(\mathcal{Z}_{11}^*, * \text{ mod } 11)$ since

$$\{\ 2^i \text{ mod } 11 \mid 1 \leq i \leq 10\} = \{2, 4, 8, 5, 10, 9, 7, 3, 6, 1\}$$

- Note that order(1) $= 1$

Order of an Element

- **The order of an element** a in an additive group is the smallest integer k such that $[k]\, a = 0$ (where 0 is the zero element of the group)
- order(3) in $(\mathcal{Z}_{11}, + \bmod 11)$ is computed by finding the smallest k such that $[k]\, 3 = 0$, which is obtained by successively computing

$$3 = 3, \quad 3+3 = 6, \quad 3+3+3 = 9, \quad 3+3+3+3 = 1, \quad \cdots$$

  until we obtain the zero element
- If we proceed, we find order(3) $= 11$ in $(\mathcal{Z}_{11}, + \bmod 11)$

$$\{\ [i]\, 3 \bmod 11 \mid 1 \leq i \leq 11\} = \{3, 6, 9, 1, 4, 7, 10, 2, 5, 8, 0\}$$

- Note that order(0) $= 1$

Lagrange's Theorem

- Theorem: The order of an element divides the order of the group.
- Lagrange's theorem applies to any group, and any element in the group
- The order of the group $(\mathcal{Z}_{11}^*, * \text{ mod } 11)$ is equal to 10, while order(3) $= 5$ in $(\mathcal{Z}_{11}^*, * \text{ mod } 11)$, and 5 divides 10, i.e., $5|10$
- order(2) $= 10$ in $(\mathcal{Z}_{11}^*, * \text{ mod } 11)$, and 10 divides 10, i.e., $10|10$
- Similarly, order(1) $= 1$ in $(\mathcal{Z}_{11}^*, * \text{ mod } 11)$, and 1 divides 10, i.e., $1|10$
- Since the order of the group $(\mathcal{Z}_{11}^*, * \text{ mod } 11)$ is 10, and the divisors of 10 are 1, 2, 5, and 10, the element orders can only be 1, 2, 5, or 10

Lagrange Theorem

- On the other hand, order(3) $= 11$ in $(\mathcal{Z}_{11}, +$ mod $11)$, and $11|11$
- Similarly, order(2) $= 11$ in $(\mathcal{Z}_{11}, +$ mod $11)$
- We also found order(0)=1
- Since the order of the group $(\mathcal{Z}_{11}, +$ mod $11)$ is 11, and 11 is a prime number (divisors are 1 and 11), the order of any element in this group can be either 1 or 11
- It turns out 0 is the only element in $(\mathcal{Z}_{11}, +$ mod $11)$ whose order is 1; all other elements have the same order 11 which is the group order

Primitive Elements

- An element whose order is equal to the group order is called **primitive**
- The order of the group $(\mathcal{Z}_{11}^*, * \bmod 11)$ is 10 and order$(2) = 10$, therefore, 2 is a primitive element of the group
- order$(2) = 11$ and order$(3) = 11$ in $(\mathcal{Z}_{11}, + \bmod 11)$, which is the order of the group, therefore 2 and 3 are both primitive elements — in fact all elements of $(\mathcal{Z}_{11}, + \bmod 11)$ are primitive except 0
- Theorem: The number of primitive elements in $(\mathcal{Z}_p^*, * \bmod p)$ is $\phi(p-1)$
- There are $\phi(10) = 4$ primitive elements in $(\mathcal{Z}_{11}^*, * \bmod 11)$, they are: 2, 6, 7, 8; all of these elements are of order 10

Cyclic Groups and Generators

- We call a group **cyclic** if all elements of the group can be generated by repeated application of the group operation on **a single element**
- This element is called a **generator**
- Any primitive element is a generator
- For example, 2 is a generator of $(\mathcal{Z}_{11}^*, * \bmod 11)$ since

$$\{2^i \mid 1 \leq i \leq 10\} = \{2, 4, 8, 5, 10, 9, 7, 3, 6, 1\} = \mathcal{Z}_{11}^*$$

- Also, 2 is a generator of $(\mathcal{Z}_{11}, + \bmod 11)$ since

$$\{\ [i]\, 2 \bmod 11 \mid 1 \leq i \leq 11\} = \{2, 4, 6, 8, 10, 1, 3, 5, 7, 9, 0\} = \mathcal{Z}_{11}$$