Pollard's Rho Algorithm for Elliptic Curves

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Pollard's Rho Algorithm

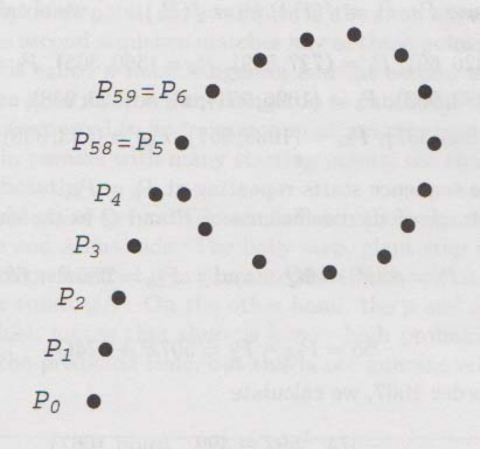
Consider the elliptic curve E over $\mathbb{F}_{2^k}$, where $|E| = n$.

Assume we want to solve the elliptic curve discrete logarithm problem: find k in $Q = kP$.

Pollard's Rho Algorithm

- Partition E into $S_1 \cup S_2 \cup S_3$, where the S_i are similar in size.
- Choose $A_i \in E$ as some scalar multiple of P.
- Let $A_{i+1} = f(A_i) = \begin{cases} A_i + P, A_i \in S_1, \\ 2A_i, A_i \in S_2, \\ A_i + Q, A_i \in S_3. \end{cases}$

Pollard's Rho Algorithm



- Image credit: Washington [1]

Pollard's Rho Algorithm

The terms of the sequence then take the form $A_i = a_j P + b_j Q$.

Once we see an equality $A_{i_1} = A_{i_2}$, we have

$$a_{j_1} P + b_{j_1} Q = a_{j_2} P + b_{j_2} Q,$$

which means that

$$\frac{a_{j_1} - a_{j_2}}{b_{j_2} - b_{j_1}} P = Q.$$

The ECDLP can thus be solved provided that $\gcd(b_{j_2} - b_{j_1}, n) = 1$.

Pollard's Rho Algorithm

- In fact, even if $\gcd(b_{j_2} - b_{j_1}, n) = d > 1$, we can compute

$$\frac{a_{j_1} - a_{j_2}}{b_{j_2} - b_{j_1}} \pmod{N/d}.$$

- There are then d possibilities for k, which is only intractable for large d.
- In practice, however, d is quite small, especially if E is chosen so that n is prime.

Pollard's Rho Algorithm

Unlike Baby-Step Giant-Step, only $O(1)$ space complexity is required:

Start with the ordered pair (A_1, A_2). Given (A_i, A_{2i}), we can compute $(A_{i+1}, A_{2i+2}) = (f(A_i), f(f(A_{2i})))$.

Pollard's Rho Algorithm

Why does this find a match?

- Suppose $A_i = A_j$. Then $A_{i+k} = A_{j+k}$ for all $k \geq 0$.
- For $k = j - 2i (\geq 0)$, we have $A_{i+j-2i} = A_{j+j-2i}$, or $A_{j-i} = A_{2(j-i)}$.
- Note that $j - i \geq i$ by construction since $j \geq 2i$.

Performance Issues

- However, it turns out that this function f performs approximately 33% more slowly than the expectation.
- It can be shown that the tail and cycle length both have an expectation of $\sqrt{\pi n/8}$.
- Therefore, a cycle should be detected within $2\sqrt{\pi n/8} = \sqrt{\pi n/2}$ iterations.

Increasing Number of Partition Elements

- Research has indicated that using more than 3 partition elements improves the randomness of the function f.
- This improves the performance of the algorithm.

Increasing Number of Partition Elements

In order to do this, we can hash the points $(x, y) \in E$ to the set $\{1, \ldots, m\}$.

- It turns out hashing based on the x-coordinate is just as effective as using the y-coordinate.
- Since the x-coordinate is a polynomial, we can represent it as a binary vector and view it as an integer for the purposes of hashing.
- We then partition evenly into m subsets of size $\frac{2^k}{m}$.

Increasing Number of Partition Elements

- We define $M_j = a_j P + b_j Q$, where the $a_j's$ and $b_j's$ are randomly chosen modulo n.
- We then define $f(A_i) = A_i + M_j$ when $A_i \in S_j$.

Increasing Number of Partition Elements

- The best choice for m in simulating a random function f seems to be in the range $[20, 30]$.
- However, there is evidence that for m around 60, the function f performs more efficiently than a random map by about 6%.

Future Work

- Collect statistics for curves over larger binary fields (the data gathered was for curves over $\mathbb{F}_{2^8}$).
- Perform similar analysis for curves over $\mathbb{F}_p$.

References

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- P. Flajolet and A. Odlyzko, Random Mapping Statistics. In *Advanced in Cryptology—EUROCRYPT '89 (Houthalen, 1989)*, volume 434 of *Lecture Notes in Comput. Sci.*, pages 329-354. Springer, Berlin, 1990.
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